$$k^2 = \omega^2\mu'\epsilon'_{\text{eff}} = \omega^2\mu'\epsilon'\left(1 + i\frac{\sigma'}{\omega\epsilon'}\right)$$

$$\tau = 1 - \frac{c^2}{c_0^2} = 1 - \frac{\mu_0\epsilon_0}{\mu'\epsilon'}$$

$$Z_{mn}{}^e = \frac{1}{\omega\epsilon'}\left[\pm(k^2 - k_{mn}{}^2)^{1/2} + i\frac{\mu'\sigma'v}{2}\right]. \tag{75}$$

Case II. TE Modes

$$E_x{}^m = -B\frac{n\pi}{b}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{i\Gamma_{mn}z} \tag{76}$$

$$E_y{}^m = B\frac{m\pi}{a}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{i\Gamma_{mn}z} \tag{77}$$

$$\boldsymbol{H}_t{}^m = Y_{mn}{}^m(\hat{z} \times \boldsymbol{E}^m) \tag{78}$$

$$H_z{}^m = \frac{k_{mn}{}^2}{i\omega\mu'}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{i\Gamma_{mn}z}, \tag{79}$$

where

$$Y_{mn}{}^m = \frac{1}{\omega\mu'}\left[\pm(k^2 - k_{mn}{}^2)^{1/2} - i\frac{\mu'\sigma'v}{2}\right]. \tag{80}$$

The other constants occurring in (75)–(79) are defined by (74).

Conclusion

The authors felt it worthwhile to include some of Minkowski's work which led to (16)–(19). The authors are convinced of the futility of trying to describe constitutive parameters μ, ϵ and σ of media in motion. As Minkowski realized, only those parameters in a medium at rest μ', ϵ' and σ', have physical meaning.

With the aid of the Maxwell-Minkowski Equations, we have derived and solved the wave equations for the electric or the magnetic field, pertaining to the guided waves in a circular or rectangular waveguide. The solution is facilitated by the introduction of vector and scalar potential-functions associated with this problem. The results demonstrate that for a moving medium, the fields, to the first-order of v/c differ in only two respects from the fields obtained when the medium is at rest. First, the propagation constant is modified by a term which depends upon the velocity as well as the constitutive constants of the stationary media, but is independent of the cross section of the guide; second, the transverse-wave impedance or admittance is also modified by a term which is independent of the dimension of the guide.

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Generalized Solutions for Optical Maser Amplifiers

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Abstract—The optical maser amplifier is treated from the transient analysis point of view using the Laplace transform method as opposed to the conventional sinusoidal steady-state analysis that sometimes leads to inconsistent results especially for the region beyond threshold. Firstly, the wave equations are expressed in terms of Laplace transforms, and then the generalized solutions for both the transmission and the reflection mode of operation are derived taking the transient terms into account. Finally, the inverse Laplace transforms are obtained yielding the generalized solutions in terms of real-time functions. In order to emphasize the point of the argument and also to compare the results of the usual sinusoidal steady-state analysis, use is made of the simplest possible model of a one-dimensional system consisting of three media, air, active medium, and air. An incident coherent transverse electromagnetic wave, which falls normally on the surface of the system, is assumed. The generalized solutions derived agree, in the region below threshold, exactly with that of the sinusoidal steady-state analysis obtained previously by other investigators. However, for the region beyond critical threshold, the generalized solutions indicate that the device goes into a state of self-oscillation with oscillation frequencies that strictly coincide with those of the Fabry-Perot type resonator. Thus, the limitation of applicability of the conventional sinusoidal steady-state analysis is clarified. Some remarks are also given on the design problem of optical maser amplifiers in connection with the transient terms involved.

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Introduction

To the authors' knowledge, most of the theoretical treatments of an optical maser amplifier reported so far have been based on the sinusoidal steady-state analysis. The investigations by Jacobs, et al. [1], [2] are typical of those approaches in which the optical maser amplifier is treated as a transmission-line or boundary-value problem in electromagnetic theory under sinusoidal steady-state conditions by replacing the time derivatives d/dt by $j\omega$. Although some insight can be gained from such a treatment, the analysis leads to some unreasonable conclusions, especially for the region beyond threshold. That is, the device still acts as an amplifier with finite and specific amplification characteristics even in the region beyond critical oscillation threshold. It is a well-known fact, however, that, if the gain per transit overcomes the total loss per transit, then feedback amplifiers such as optical masers can no longer behave as an amplifier and, instead, become oscillators whose performance is substantially independent of the input signal waves.

The purpose of the present paper is to derive the generalized solution of the optical maser amplifier, taking the transient terms into account, and to point out the limitation of applicability of sinusoidal steady-state analysis. In the following analysis, the optical maser amplifier is conceived as a multilayer structure and is treated on the transient theory point of view with the Laplace transform method, replacing the time derivatives d/dt by complex frequency s as opposed to the conventional sinusoidal steady-state analysis. In order to emphasize the essential point of the argument and also to compare the results of sinusoidal steady-state analysis, use is made of the simplest possible model of the one-dimensional system consisting of three media, air, active medium, and air. The transverse electromagnetic wave is assumed as an incident coherent radiation that falls normally on the surface of the system. It is possible, however, to extend the present analysis to somewhat more complicated systems, such as for instance, the five-layer structure consisting of air, reflector, active medium, reflector, and air.

Theory

Laplace Transforms of Wave Equations

A transverse electromagnetic wave (TEM) propagating the direction of the z axis in Cartesian coordinates is described by the following one-dimensional scalar wave equations:

$$\frac{\partial E}{\partial z} = -\mu\frac{\partial H}{\partial t}$$

$$\frac{\partial H}{\partial z} = -\sigma E - \epsilon\frac{\partial E}{\partial t} \tag{1}$$

where ϵ, μ, and σ are, respectively, the permittivity, permeability, and conductivity of the medium through which the TEM wave propagates. $E(z, t)$ represents the x component of the electric field, whereas $H(z, t)$ denotes the y component of the magnetic field of the wave in question.

Assuming the initial distribution of the wave as

$$E(z, 0) = 0$$

$$H(z, 0) = 0 \tag{2}$$

and introducing the complex frequency $s = \xi + j\omega$, the Laplace transforms of the wave equation (1) becomes

$$\frac{\partial F}{\partial z} = -\mu s U$$

$$\frac{\partial U}{\partial z} = -\sigma(s)F - \epsilon s F \tag{3}$$

where $F(z, s)$ and $U(z, s)$ have been defined as Laplace transforms of $E(z, t)$ and $H(z, t)$, respectively. Partial-differential equations for F and U alone can be yielded from the foregoing equations:

$$\frac{\partial^2 F}{\partial z^2} = \Gamma^2 F$$

$$\frac{\partial^2 U}{\partial z^2} = \Gamma^2 U \tag{4}$$

where

$$\Gamma = \sqrt{\mu s\{\epsilon s + \sigma(s)\}} \tag{5}$$

and in particular, if $|\sigma(s)| \ll |\epsilon s)|$,

$$\Gamma = s\sqrt{\epsilon\mu} + \frac{\sigma(s)}{2}\sqrt{\frac{\mu}{\epsilon}} \equiv \frac{s}{v} - \alpha(s) \tag{6}$$

where $v = 1/\sqrt{\epsilon\mu}$ is a velocity of propagation of plane wave in the medium. The general solutions for (4) are

$$F = C_1 e^{-\Gamma z} + C_2 e^{\Gamma z}$$

$$U = (1/Z)(C_1 e^{-\Gamma z} - C_2 e^{\Gamma z}) \tag{7}$$

in which we have defined

$$Z = \sqrt{\frac{\mu S}{\epsilon s + \sigma(s)}}\,. \tag{8}$$

C_1 and C_2 in (7) are arbitrary constants that will be determined by boundary conditions. Again, if the conductivity of the medium is so small that $|\sigma(s)| \ll |\epsilon s|$, (8) reduces simply to

$$Z = \sqrt{\mu/\epsilon}. \tag{9}$$

Let the values of F and U at $z = l$ be F_b and U_b, respectively. Then, the arbitrary constants C_1 and C_2 in (7) are specified, and, in turn, the expressions for F and U at $z = 0$, say F_a and U_a, can be written in the form

$$F_a = F_b \cosh \Gamma l + Z U_b \sinh \Gamma l$$

$$U_a = (1/Z) F_b \sinh \Gamma l + U_b \cosh \Gamma l. \tag{10}$$

Generalized Solutions for Optical Maser Amplifier

Let us consider now a simplified model of the optical maser amplifier consisting of three layers as shown in Fig. 1, each layer being infinite in both x and y planes. Regions I and III are air, whereas region II is assumed to be a linear, isotropic, and homogeneous active medium having uniformly distributed negative conductance. The incident electromagnetic plane wave

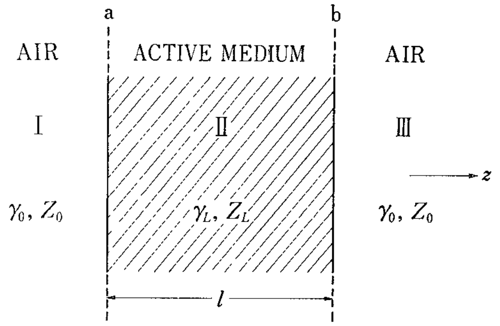

Fig. 1. One-dimensional model of optical maser amplifier consisting of three media, air, active medium, and air.

travels from region I through III along the z axis, i.e., normally to the planar surfaces of each layer. The wave suffers a reflection and transmission at both boundary surfaces a and b and multiple internal reflections and amplification in the active region II. Thus, the transmission system indicated in Fig. 1 is equivalent to the one-dimensional model of a Fabry-Perot interferometer with an active medium inserted between the pair of opposing reflecting surfaces.

In regions I and III, where $\sigma=0$, (5) and (8) become

$$\Gamma_0 = s\sqrt{\epsilon_0\mu_0} = s/c$$
$$Z_0 = \sqrt{\mu_0/\epsilon_0} \tag{11}$$

where ϵ_0 and μ_0 are the permittivity and permeability of air, respectively, and $c=1/\sqrt{\epsilon_0\mu_0}$ represents the velocity of light in the air.

On the other hand, the conductivity σ in the active medium of region II can be described in the following form [3]:

$$\sigma(j\omega) = j\omega\chi(j\omega) \tag{12}$$

where

$$\chi(j\omega) = \frac{\chi_0}{\omega - \omega_0 - j\Delta\omega}$$

where it is assumed that the spectral line of stimulated emission of radiation has a Lorentzian shape with full width at half-maximum of $2\Delta\omega$ centered about angular frequency ω_0. χ_0 is a positive constant whose magnitude depends on the intensity of the pump power supplied into the active medium.

Replacing $j\omega$ by complex frequency s, the Laplace transform of (12) becomes

$$\sigma(s) = s\chi(s) \tag{13}$$

where

$$\chi(s) = \frac{-\chi_0}{\omega_0 + js + j\Delta\omega}.$$

Thus, in the region II,

$$\Gamma_L = s\sqrt{\mu\{\epsilon + \chi(s)\}}$$

$$Z_L = \sqrt{\frac{\mu}{\epsilon + \chi(s)}} \tag{14}$$

where ϵ and μ refer to the permittivity and permeability of the active medium, respectively. Since, in practice, $|\chi(s)|$ can be regarded much smaller than $|\epsilon|$, (14) can be written in the form similar to (6) and (9):

$$\Gamma_L = \frac{s}{v} - \alpha(s)$$
$$Z_L = \sqrt{\frac{\mu}{\epsilon}} \tag{15}$$

where

$$\alpha(s) = \frac{\sigma(s)}{2}\sqrt{\frac{\mu}{\epsilon}} = -\frac{s\chi(s)}{2}\sqrt{\frac{\mu}{\epsilon}}. \tag{16}$$

It should be noted that the real part of $\alpha(s)$ in (15) is positive in sign and hence is referred to the amplification constant per unit length in an active medium.

In order to formulate the boundary-value problem, let us define the following notations: $F_i(s)$ and $U_i(s)$ for the Laplace transforms of electric and magnetic fields of incident wave traveling in region I toward region II, $F_r(s)$ and $U_r(s)$ for that of reflected wave reflecting from boundary a, and $F_t(s)$ and $U_t(s)$ for that of transmitted wave leaving from boundary b. It will be assumed further that the region I and III are semi-infinite in the z direction so that there would be no reflected waves in both regions. Then, the Laplace transforms of electric and magnetic fields at boundary a, say $F_a(s)$ and $U_a(s)$, respectively, can be expressed as

$$F_a = F_i + F_r$$
$$U_a = U_i + U_r. \tag{17}$$

Similarly, at boundary b,

$$F_b = F_t, \qquad U_b = U_t \tag{18}$$

In addition,

$$\frac{F_i}{U_i} = \frac{F_t}{U_t} = -\frac{F_r}{U_r} = Z_0 \tag{19}$$

where Z_0 is the intrinsic impedance in free space given by (11).

On the other hand, referring to (10),

$$F_a = F_b\cosh\Gamma_L l + Z_L U_b \sinh\Gamma_L l$$
$$U_a = (1/Z_L)F_b\sinh\Gamma_L l + U_b\cosh\Gamma_L l \tag{20}$$

where l is the length of the active region II as shown in Fig. 1. Substituting (17), (18), and (19) into (20),

$$F_i + F_r = F_t\cosh\Gamma_L l + (Z_L/Z_0)F_t\sinh\Gamma_L l$$
$$F_i - F_r = (Z_0/Z_L)F_t\sinh\Gamma_L l + F_t\cosh\Gamma_L l \tag{21}$$

from which

$$\frac{F_t}{F_i} = \frac{1}{\cosh\Gamma_L l + A\sinh\Gamma_L l} \equiv Y_t(s) \tag{22}$$

$$\frac{F_r}{F_i} = \frac{B \sinh \Gamma_L l}{\cosh \Gamma_L l + A \sinh \Gamma_L l} \equiv Y_r(s) \tag{23}$$

where

$$A = \frac{1}{2}\left(\frac{Z_L}{Z_0} + \frac{Z_0}{Z_L}\right) > 1$$

$$B = \frac{1}{2}\left(\frac{Z_L}{Z_0} - \frac{Z_0}{Z_L}\right). \tag{24}$$

Note that, if $Z_0 > Z_L$,

$$B = -\sqrt{A^2 - 1}\ . \tag{25}$$

The poles of the functions $Y_t(s)$ and $Y_r(s)$ defined by (22) and (23), respectively, are given by

$$s_n = 2(\alpha_n l - L_r)(\delta f) + j\omega_n \tag{26}$$

where

$$\omega_n = n\pi(v/l), \qquad \delta f = (v/2l),$$

$$\alpha_n = \operatorname{Re} \alpha(s_n), \qquad L_r = \ln \left| \frac{Z_0 + Z_L}{Z_0 - Z_L} \right|$$

and n is integer. Re $\alpha(s_n)$ denotes the real part of $\alpha(s_n)$, and L_r represents the reflection loss at boundary surface a or b. [See (47), (48), and (50) in Appendix.]

Now, the inverse Laplace transforms of $F_t(S)$ and $F_r(s)$ given by (22) and (23), respectively, yield the transmitted and reflected electric fields in terms of the function of time:

$$E_t(t) = \frac{1}{2\pi j} \int_{\xi - j\infty}^{\xi + j\infty} F_i(s) Y_t(s) e^{st} ds$$

$$E_r(t) = \frac{1}{2\pi j} \int_{\xi - j\infty}^{\xi + j\infty} F_i(s) Y_r(s) e^{st} ds. \tag{27}$$

If the incident electric field is assumed to be

$$E_i(t) = \begin{cases} 0, & t < 0 \\ E_0 e^{j\omega t}, & t > 0. \end{cases} \tag{28}$$

$F_i(s)$, the Laplace transform of $E_i(t)$, is simply given by

$$F_i(s) = \frac{E_0}{s - j\omega}\ . \tag{29}$$

Finally, substituting (22), (23), and (29) into (27) and performing the evaluation of the inverse Laplace integrals, we obtain the generalized solutions for the three-layer optical maser amplifier involving the transient terms as follows (see Appendix):

$$E_t(t) = \frac{E_0 e^{j\omega t}}{\cosh \gamma_L l + A \sinh \gamma_L l} + \sum_n (-1)^n E_n \exp\left[2(\alpha_n l - L_r)(\delta f) t + j\omega_n t\right] \tag{30}$$

$$E_r(t) = \frac{B E_0 e^{j\omega t} \sinh \gamma_L l}{\cosh \gamma_L l + A \sinh \gamma_L l} + \sum_n E_n \exp\left[2(\alpha_n l - L_r)(\delta f) t + j\omega_n t\right] \tag{31}$$

where

$$\gamma_L = j\omega\sqrt{\mu\{\epsilon + \chi(j\omega)\}} \simeq j\frac{\omega}{v} + \frac{j\omega\chi(j\omega)}{2}\sqrt{\frac{\mu}{\epsilon}}$$

$$\equiv j\beta - \alpha(j\omega)$$

$$E_n = \frac{E_0}{|B|\, \Gamma'(s_n) l \{2(\alpha_n l - L_r)(\delta f) + j(\omega_n - \omega)\}}$$

$$\Gamma'(s_n) = \left.\frac{d\Gamma(s)}{ds}\right|_{s=s_n}.$$

Solutions in Stable Region

If the magnitude of amplification of the wave per round trip of propagation through the active medium is smaller than the total reflection loss, the system would be stable. In such a case, $\alpha_n l - L_r < 0$, so that the second terms in both (30) and (31) vanish in the limit when the time t approaches infinity; hence, the transmitted and reflected electric fields can be expressed only by the non-vanishing first terms alone:

$$E_t(t) = \frac{E_0 e^{j\omega t}}{\cosh \gamma_L l + A \sinh \gamma_L l} \tag{32}$$

$$E_r(t) = \frac{B E_0 e^{j\omega t} \sinh \gamma_L l}{\cosh \gamma_L l + A \sinh \gamma_L l}\ . \tag{33}$$

The foregoing expressions strictly agree with Jacobs' sinusoidal steady-state solutions as would be expected since all the transient terms should vanish at steady state.

In order to discuss briefly the amplification characteristics of the device in such a stable region, let us assume the following approximations for simplicity:

$$\gamma_L = j\beta - \alpha(j\omega) \simeq j\beta - \alpha \tag{34}$$

where

$$\alpha = \operatorname{Re} \alpha(j\omega) = \frac{\chi_0}{2}\sqrt{\frac{\mu}{\epsilon}} \frac{\Delta\omega}{(\omega - \omega_0)^2 + (\Delta\omega)^2}\ .$$

Then, the power gain in the transmission mode of operation can be found from (32) using (34):

$$\left|\frac{E_t}{E_i}\right|^2 = \frac{1}{(\cosh \alpha l - A \sinh \alpha l)^2 + B^2 \sin^2 \beta l}\ . \tag{35}$$

Solving $d/dl|E_t/E_i|^2 = 0$ under the assumption of $\alpha \ll \beta$, we have

$$\beta l = n(\pi/2) \quad \text{or} \quad l = n(\lambda/4) \tag{36}$$

where n is integer. The transmission gain becomes maxima for even number of n in (36). That is,

$$\beta l = m\pi \tag{37}$$

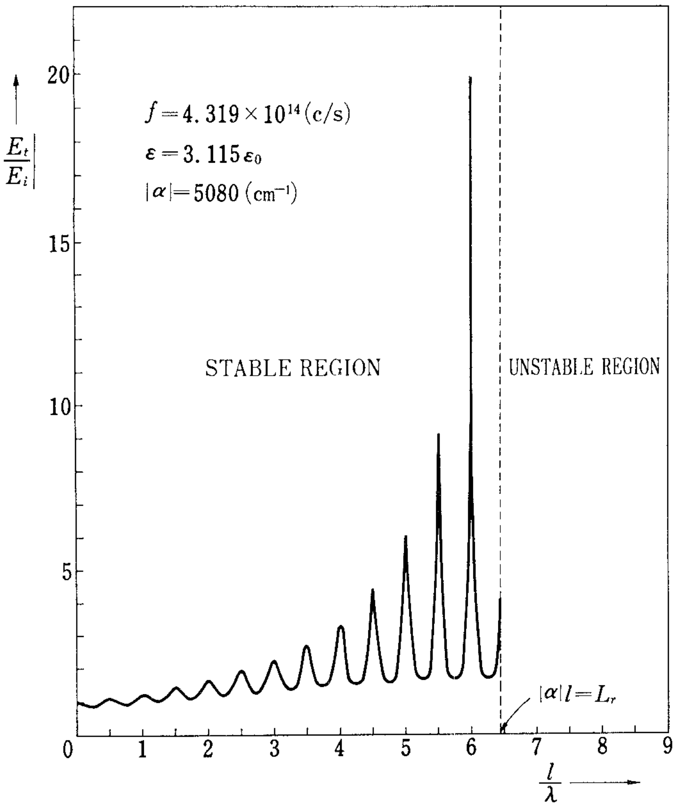

Fig. 2. The magnitude of amplitude amplification for a three-layer optical maser amplifier in the transmission mode of operation.

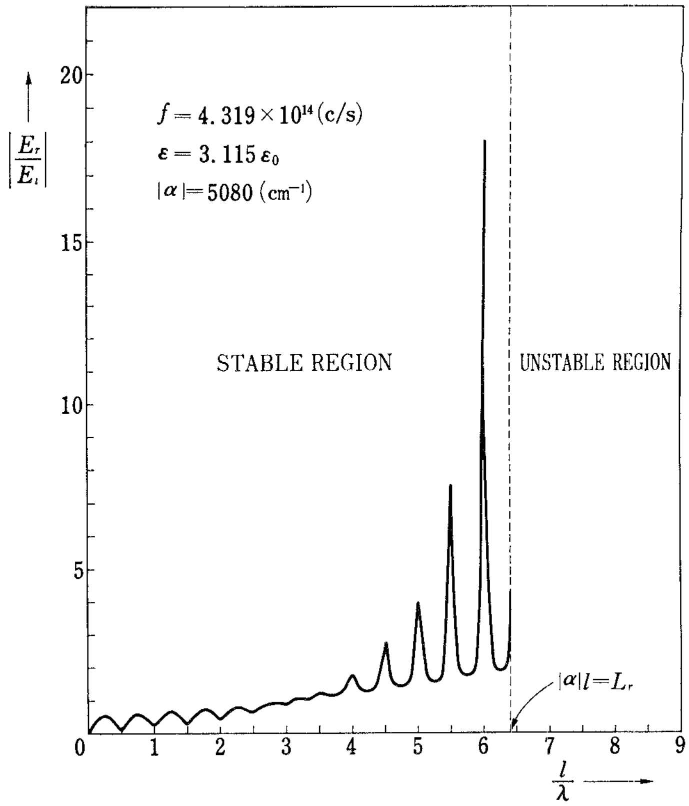

Fig. 3. The magnitude of amplitude amplification for a three-layer optical maser amplifier in the reflection mode of operation.

where m is integer. The minima of transmission gain would occur for odd number of n. Namely,

$$\beta l = (2m + 1)(\pi/2). \tag{38}$$

On the other hand, the power gain in the reflection mode of operation is yielded from (33) using (34):

$$\left|\frac{E_r}{E_i}\right|^2 = 1 + \frac{A \sinh 2\alpha l - \cosh 2\alpha l}{(\cosh \alpha l - A \sinh \alpha l)^2 + B^2 \sin^2 \beta l} \cdot \tag{39}$$

The particular length of l for which $d/dl|E_r/E_i|^2 = 0$ is also given by (36) under the same assumption. It should be noted, however, that, if $0 < \alpha l < \frac{1}{2} \coth^{-1} A = L_r/2$, the numerator of the second term on the right-hand side of (39) becomes negative, and hence the minima of the reflection gain will occur at $\beta l = m\pi$ and the maxima at $\beta l = (2m+1)\pi/2$. For $L_r/2 < \alpha l < L_r$, to the contrary, $A \sinh 2\alpha l - \cosh 2\alpha l > 0$, so that the condition on l becomes entirely the same as that for the case of the transmission mode of operation. That is, the reflection gain becomes maxima at $\beta l = m\pi$ and minima at $\beta l = (2m+1)\pi/2$.

Let us illustrate by a numerical example using the following parameters:

$$f = 4.319 \times 10^{14} \text{ (c/s)}$$

$$\epsilon = 3.115\epsilon_0$$

$$\alpha = 5080 \text{ (cm}^{-1}\text{)}.$$

The calculated amplification characteristics for transmission and reflection type of operation are shown in Figs. 2 and 3, respectively. In this particular example, the critical threshold for oscillation is given by $l/\lambda = 6.42$ at which the gain per transit becomes equal to the loss per transit, i.e., $\alpha_n l = L_r$. Note that the discussion given in this section is restricted only within the stable region below this critical threshold. The foregoing analysis, together with the numerical example, indicates that, in order to achieve the maximum gain, the length of an active medium should be designed as long as possible within the region below threshold, satisfying the maximization condition just stated.

It is also to be pointed out that, if the length l is chosen too close to the critical threshold so that $\alpha_n l - L_r \simeq 0$, a time rate of decay of transient terms would become so slow that somewhat undesirable effects against amplification of the signal may become appreciable.

Solutions in Unstable Region

In the region above critical threshold for which $\alpha_n l - L_r > 0$, the system becomes unstable since the amplification gained per transit overcomes the reflection loss per transit. More strictly, the second terms in both (30) and (31) become infinity in the limit when time t approaches infinity so that the magnitudes of solutions for both transmitted and reflected waves become infinity. Practically, however, the magnitudes of the fields never increase up to infinity even in the limit of $t \rightarrow \infty$ because of the saturation of the gain which is due to the nonlinearity of the interaction between fields and matter. Instead, the system maintains a self-oscillation with finite amplitude $E_{n,\mathrm{osc}}$, whose magnitude depends on the nonlinear characteristic of an active medium involved which is beyond the scope of our discussion. For such a final steady-oscillation state, the (nonlinear) gain $\alpha_n l$ becomes equal to the loss L_r, and hence the second terms in (30) and (31), respectively, can be expressed in the following form:

$$E_{t,\mathrm{osc}} = \sum_{n=i}^{k} (-1)^n E_{n,\mathrm{osc}} e^{j\omega_n t}$$

$$E_{r,\mathrm{osc}} = \sum_{n=i}^{k} E_{n,\mathrm{osc}} e^{j\omega_n t}$$

where both i and k are the particular integral number of n in the sense that, for $i \leq n \leq k$, the oscillation condition $\alpha_n l - L_r \geq 0$ is satisfied.

The oscillation frequencies f_n or the oscillation wavelengths λ_n are given by

$$f_n = \frac{\omega_n}{2\pi} = n \frac{v}{2l} = n(\delta f) \quad \text{or} \quad \lambda_n = \frac{2l}{n} \tag{40}$$

where n is integer. Equation (40) just coincides with well-known oscillation frequencies in the Fabry-Perot laser resonators. δf represents the separation between adjacent frequencies of the resonance. In other words, the system oscillates, in the region beyond critical threshold, with finite amplitude and specific frequencies f_n given by (40) which are entirely independent of the input signal waves.

Conversely, as long as we treat the problem on the basis of sinusoidal steady-state analysis such as, for instance, Jacobs' approach, it seems to be impossible to obtain these results and, instead, some inconsistent conclusions such as a finite and specific amplification characteristic in the unstable region is yielded, even though the theory is entirely linear.

Conclusions

The generalized solutions for the Fabry-Perot type three-layer optical maser amplifier were derived on the basis of a transient analysis point of view taking the transient terms into account. As a result, it was shown that the generalized solutions below threshold agree exactly with the sinusoidal steady-state solutions obtained so far by Jacobs et al. and that, beyond critical threshold, the device becomes unstable, resulting in the self-oscillations with particular oscillating frequencies which strictly agree with the well-known resonant frequencies of Fabry-Perot type optical masers. Some remarks were also given on the design problem of optical maser amplifiers in connection with transient terms involved.

Appendix

Evaluation of the Inverse Laplace Transforms

The inverse Laplace integrals to be evaluated are

$$E_t(t) = \frac{1}{2\pi j} \int_{\xi - j\infty}^{\xi + j\infty} F_i(s) Y_t(s) e^{st} ds \tag{41}$$

$$E_r(t) = \frac{1}{2\pi j} \int_{\xi - j\infty}^{\xi + j\infty} F_i(s) Y_r(s) e^{st} ds \tag{42}$$

where $Y_t(s)$, $Y_r(s)$, and $F_i(s)$ are given by (22), (23), and (29), respectively. The evaluation of the integrals in (41) and (42) are performed by utilizing the Cauchy fundamental integral theorem (see, e.g. [4]) based on the theorem of sum of residues taking a closed contour $ABCDA$ on the complex plane s as shown in Fig. 4. The path is selected in such a manner that the integrands of both (41) and (42) be regular within the region of Re $s \geq \xi$. Furthermore, η_p is chosen as

$$\eta_p = (2p + 2q + 1)(\delta f)\pi \tag{43}$$

where p is integer, q is integer satisfying $q \gg L_r/\pi$ and also $q \gg \omega/2\pi(\delta f)$, so that the contour does not pass into or over singularities of the integrands.

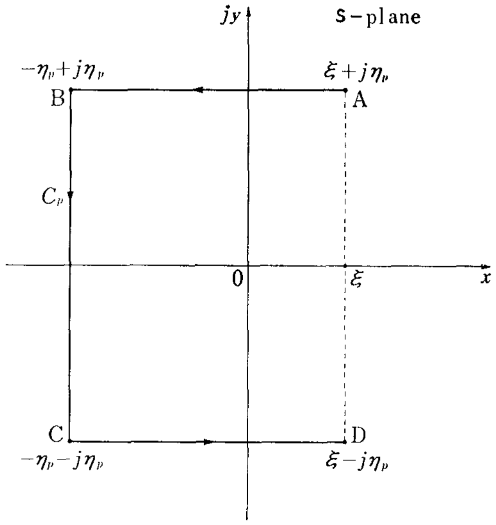

Fig. 4. Contour taken on the complex plane s used in computing the inverse Laplace integrals.

Then, it can be shown that the contributions of the line integral along a path $ABCD$, say C_p, indicated by a solid line in Fig. 4, vanish in the limit when η_p approaches infinity:

$$\lim_{\eta_p \rightarrow \infty} \int_{C_p} F(s) Y(s) e^{st} ds \rightarrow 0. \tag{44}$$

Next, the residue for the pole $j\omega$ of $F_i(s)$ in (41) is given as

$$\phi_t = E_0 Y_t(j\omega) e^{j\omega t}. \tag{45}$$

The pole s_n of $Y_t(s)$ is determined by

$$\coth \Gamma_L(s_n) l = -A. \tag{46}$$

Rewrite (46),

$$\exp\left[2\Gamma_L(s_n)l\right] = \frac{A-1}{A+1} = \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right)^2 \equiv R^2. \qquad (47)$$

R, defined by (47), represents the reflection coefficient at the boundary surface a or b. Therefore, if we set as

$$R = e^{-L_r} \qquad (48)$$

then L_r denotes the reflection loss at those boundary surfaces. Using the foregoing relation, (47) becomes

$$\Gamma(s_n)l = -L_r + jn\pi. \qquad (49)$$

Remembering that, in the most practical cases, $\epsilon \gg |\chi(s)|$, the pole s_n can be written, with the aid of (15), as

$$s_n = 2(\alpha_n l - L_r)(\delta f) + j\omega_n. \qquad (50)$$

The residue for this pole S_n is

$$\rho_t = \frac{(-1)^n E_0 e^{s_n t}}{|B|\,\Gamma'(s_n)l(s_n - j\omega)}\cdot \qquad (51)$$

Similarly, the residue for the pole of $F_t(s)$ in (42) is given by

$$\phi_r = E_0 Y_r(j\omega)e^{j\omega t} \qquad (52)$$

and for the pole of $Y_r(s)$,

$$\rho_r = \frac{E_0 e^{s_n t}}{|B|\,\Gamma'(s_n)l(s_n - j\omega)}\cdot \qquad (53)$$

According to the Cauchy fundamental integral theorem, the closed line integral along a closed contour $ABCDA$ can now be given by the sum of all these residues enclosed. On the other hand, the contributions along a path C_p vanish in the limit when η_p approaches infinity as shown in (44). Accordingly,

$$\lim_{\eta_p \to \infty} \oint F(s)Y(s)e^{st}ds = \int_{\xi - j\omega}^{\xi + j\infty} F(s)Y(s)e^{st}ds$$
$$= 2\pi j \Sigma(\phi + \rho). \qquad (54)$$

Finally, substituting (45) and (51) or (52) and (53) into the foregoing equation, the transmitted and reflected electric fields can be expressed in terms of the function of time as follows:

$$E_t(t) = E_0 Y_t(j\omega)e^{j\omega t} + \sum_n (-1)^n E_n e^{s_n t} \qquad (55)$$

$$E_r(t) = E_0 Y_r(j\omega)e^{j\omega t} + \sum_n E_n e^{s_n t}. \qquad (56)$$

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Characteristics of Loaded Rectangular Waveguides

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***Abstract*—Electromagnetic fields of a rectangular waveguide with an arbitrarily loaded slab are theoretically analyzed. Eigenvalues for the transmission modes are presented in the form of universal eigenvalue charts. Electric field distributions in the loaded waveguide are obtained theoretically, and they are compared with the results of measurements. Power attenuation is also discussed, and attenuation charts that give the lowest limitation for the attenuation are shown. As an example of application the attenuation characteristics of waveguide resistance attenuators are investigated, and a new interpretation is derived for the phenomena where the curves of attenuation characteristics have sharp peak points.**

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I. Introduction

THE TRANSMISSION characteristics of a rectangular waveguide loaded with a dielectric slab or resistive strip at the center of the waveguide parallel to the electric field have already been analyzed by several investigators [1]–[4]. However, the characteristics for a general case where the slab with an arbitrary admittance is loaded at a place with various distances from the side wall have not been given [5].

The purpose of this paper is to present such characteristics of the waveguides with some charts that show the dependence of the characteristics to the various parameters.